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# **Radial Point** Interpolation Collocation Method (RPICM) for Partial Differential Equations

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Abstract—This paper presents a truly meshfree method referred to as radial point interpolation collocation method (RPICM) for solving partial differential equations. This method is different from the existing point interpolation method (PIM) that is based on the Galerkin weak-form. Because it is based on the collocation scheme no background cells are required for numerical integration. Radial basis functions are used in the work to create shape functions. A series of test examples were numerically analysed using the present method, including 1-D and 2-D partial differential equations, in order to test the accuracy and efficiency of the proposed schemes. Several aspects have been numerically investigated, including the choice of shape parameter c which can greatly affect the accuracy of the approximation; the enforcement of additional polynomial terms; and the application of the Hermite-type interpolation which makes use of the normal gradient on Neumann boundary for the solution of PDEs with Neumann boundary conditions. Particular emphasis was on an efficient scheme, namely Hermite-type interpolation for dealing with Neumann boundary conditions. The numerical results demonstrate that good improvement on accuracy can be obtained after using Hermite-type interpolation. The h-convergence rates are also studied for RPICM with different forms of basis functions and different additional terms. © 2005 Elsevier Ltd. All rights reserved.

Keywords—RPICM, Hermite-type interpolation, Meshfree, Partial differential equations.

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# 1. INTRODUCTION

In recent years, there has been great interest to improve meshfree methods based on radial basis functions (RBF) in the field of computational mathematics and mechanics [1–8]. However, the primary disadvantage of the traditional RBF approach is that the coefficient matrices obtained from this discretization scheme are fully populated. It will cause great inconvenience for largescale practical problems. At present, there are mainly two approaches to improve the traditional RBF approach. One is to improve the conditioning of the coefficient matrix and the solution accuracy using some special mathematical techniques. The other is to obtain banded coefficient matrices using compactly support radial basis functions. It has been shown that these two approaches cannot always produce satisfactory results. More effective approaches are therefore needed.

Point interpolation method (PIM) was developed by Liu *et al.* [9], and it has further been studied [10–15]. As the name implies, PIM obtains its approximation by letting the interpolation function pass through the function values at each scattered node within the defined local support domain. In PIM, its shape function possesses the Kronocker delta function property so that the essential boundary conditions, which have been troubling meshfree researchers for recent years, can be easily handled like in the traditional finite-element method (FEM). So far, PIM is based on Galerkin or Petrov-Galerkin weak forms, and numerical integrations are required. Like other Galerkin-based meshfree methods, the inevitable background cell must be used in integration processes. In contrast to Galerkin-based approaches, the collocation method is simple and efficient to solve partial differential equations without the need of numerical integrations. Collocation is known as an efficient and highly accurate numerical solution procedure for partial differential equations, which is similar to the MQ-RPICM, has been presented and applied to solve PDEs.

However, the research results in [17] showed that the accuracy obtained by using direct collocation scheme is a bit poor especially on boundary. In addition, the collocation scheme, which has difficulties in dealing with Neumann boundary conditions, is very different from the Galerkin scheme that can deal with Neumann boundary conditions naturally. Liszka *et al.* [17] proposed a Hermite-type interpolation scheme in Generalized Finite Difference Method (GFDM) to improve the accuracy of collocation-based approach for solving solid problems. Zhang *et al.* [7] applied Hermite-type interpolation in compactly supported radial basis function method successfully. Liu *et al.* [14] presented an efficient RPICM based on thin plate spline (TPS) for solving 2-D linear elastic problem with especially attention for dealing with force boundary condition.

In this paper, the Hermite-type interpolation is adopted in the point interpolation in order to improve the accuracy. Approximate field functions are carried out not only with the nodal values but also with the normal gradient at the Neumann boundaries by taking the advantage of the point interpolation method based on radial basis functions.

In this paper, the radial point interpolation collocation method (RPICM) is presented. The formulation for constructing shape functions based on radial point interpolation and Hermite radial point interpolation is described and formulated in Section 2 and Section 3. The detail collocation schemes are discussed in Section 4. In Section 5, the accuracy and simplicity of this presented approach is shown numerically by a series of test examples, and h-convergence of this method is numerically analysed. We conclude with a summary in Section 6.

# 2. RADIAL BASIS POINT INTERPOLATION

The approximation of a function  $u(\mathbf{x})$ , using radial basis functions, may be written as a linear combination of n radial basis functions, *viz.*,

$$u(\mathbf{x}) \cong \hat{u}(\mathbf{x}) = \sum_{i=1}^{n} a_i \phi\left( \left\| \mathbf{r} - \mathbf{r}_i \right\|, c_i \right), \tag{1}$$

where n is the number of points in the support domain near  $\mathbf{x}, a_i$  are coefficients to be determined and  $\phi$  are the MQ, or inverse-MQ, or Gaussian basis function, or thin plate spline (TPS) function.

These well-known radial basis functions are as follows.

MQ.

$$\phi\left(\left\|\mathbf{r}-\mathbf{r}_{i}\right\|,c_{i}
ight)=\left(\sqrt{\left\|\mathbf{r}-\mathbf{r}_{i}\right\|^{2}+c_{i}^{2}}
ight)^{q}.$$

GAUSSIAN BASIS FUNCTION.

$$e^{-c_i^2 \left( \| \mathbf{r} - \mathbf{r}_i \|^2 / r_c^2 
ight)}$$

THIN PLATE SPLINE.

$$\|\mathbf{r} - \mathbf{r_i}\|^{2M} \log (\|\mathbf{r} - \mathbf{r_i}\|).$$

The shape parameter can be defined as  $c_i = (\alpha_c r_c)$  [8]. Where **r** is the distance between two nodes. In 2-D problems, we have

$$\|\mathbf{r} - \mathbf{r}_i\| = \sqrt{(x - x_i)^2 + (y - y_i)^2}.$$
 (2)

The constant  $c_i$  is a shape parameter. How to choose the optimal shape parameter is a problem that has received the attention of many researchers [8,16]. So far, there is no mathematical theory developed for determining the optimal value. Detailed guidelines on how to choose these parameters can be found in Liu's recent monograph [8]. Optimal values for these parameters for PIMs based on Galerkin and Petrov-Galerkin weak forms were found via numerical experiments and provided in this book. Here, the form of dimensionless shape parameter  $\alpha_c$  will be employed and investigated for RPICM. The constant  $r_c$  is the characteristic length that is related to the nodal space in the local support domain of the collocation point and it is usually the average nodal spacing for all the nodes in this support domain. The coefficients  $a_i$  in equation (1) can be determined by enforcing that the function interpolations pass through all n nodes within the support domain.

The interpolations of a function at the  $k^{\text{th}}$  point can have the form of

$$\hat{u}(\mathbf{x}_{k}) = a_{1}\phi(\|\mathbf{r}_{k} - \mathbf{r}_{1}\|, c_{1}) + a_{2}\phi(\|\mathbf{r}_{k} - \mathbf{r}_{2}\|, c_{2}) + \ldots + a_{n}\phi(\|\mathbf{r}_{k} - \mathbf{r}_{n}\|, c_{n}), \qquad k = 1, 2..., n.$$
(3)

The function interpolation can be expressed in a matrix form as follows:

$$\hat{\mathbf{u}}^e = \mathbf{\Phi} \mathbf{a},\tag{4}$$

$$\begin{bmatrix} \phi(\|\mathbf{r}_1 - \mathbf{r}_l\|, c_l) & \cdots & \phi(\|\mathbf{r}_1 - \mathbf{r}_i\|, c_i) & \cdots & \phi(\|\mathbf{r}_1 - \mathbf{r}_n\|, c_n) \\ \vdots & \ddots & \vdots & \ddots & \vdots \end{bmatrix}$$

$$\boldsymbol{\Phi} = \left[ \begin{array}{cccc} \phi\left( \| \mathbf{r}_{i} - \mathbf{r}_{1} \|, c_{1} \right) & \cdots & \phi\left( \| \mathbf{r}_{i} - \mathbf{r}_{i} \|, c_{i} \right) & \cdots & \phi\left( \| \mathbf{r}_{i} - \mathbf{r}_{n} \|, c_{n} \right) \\ \vdots & \ddots & \vdots & \ddots & \vdots \end{array} \right] , \quad (5a)$$

$$\left[\phi\left(\left\|\mathbf{r}_{n}-\mathbf{r}_{1}\right\|,c_{1}\right)\cdots\phi\left(\left\|\mathbf{r}_{n}-\mathbf{r}_{i}\right\|,c_{i}\right)\cdots\phi\left(\left\|\mathbf{r}_{n}-\mathbf{r}_{n}\right\|,c_{n}\right)\right]_{n\times n}$$

$$\hat{\mathbf{u}}^{e} = \begin{bmatrix} \hat{u} (\mathbf{x}_{1}) & \cdots & \hat{u} (\mathbf{x}_{k}) & \dots & \hat{u} (\mathbf{x}_{n}) \end{bmatrix}^{T},$$
<sup>(30)</sup>

$$\mathbf{a} = \begin{bmatrix} a_1 & \cdots & a_i & \cdots & a_n \end{bmatrix}^{\top}. \tag{5c}$$

Thus, the unknown coefficients vector is found to be

$$\mathbf{a} - \boldsymbol{\Phi}^{-1} \hat{\mathbf{u}}^{\boldsymbol{e}}.$$
 (6)



# Collocation Point

Figure 1. The four quadrants criterion.

The form of the approximation function can be obtained as follows:

$$\hat{u}(\mathbf{x}) = \varphi \mathbf{a} = \varphi \Phi^{-1} \hat{\mathbf{u}}^e = \psi \hat{\mathbf{u}}^e, \tag{7}$$

$$\varphi = [\phi(\|\mathbf{r} - \mathbf{r}_1\|, c_1) \quad \phi(\|\mathbf{r} - \mathbf{r}_2\|, c_2) \quad \phi(\|\mathbf{r} - \mathbf{r}_n\|, c_n)]_{1 \times n},$$
(8)

where the matrix of shape functions can be expressed as follows

$$\psi = \varphi \Phi^{-1} = \begin{bmatrix} \psi_1 & \cdots & \psi_i & \cdots & \psi_n \end{bmatrix}_{1 \times n}$$
(9)

in which  $\psi_i$  (i = 1, ..., n) are shape functions for points in the support domain, which satisfy.

$$\psi_i(x_j) = \begin{cases} 1, & j = i, \\ 0, & j \neq i. \end{cases}$$
(10)

Thus, the shape functions constructed have the delta function property, which is very attractive to impose essential boundary condition in the Galerkin-based weak form meshfree methods.

Let us now examine whether  $\Phi^{-1}$  exists and how to assure its existence. To do it, a good performance arrangement of nodes in the support domain near **x** must be constructed for this collocation scheme. Here the four quadrants criterion introduced is a useful way to do it. As shown in Figure 1, point **x** is regarded as the current collocation point and four nearest points to **x** must be found in four quadrants, respectively. Their distances from **x** are  $d_1$ ,  $d_2$ ,  $d_3$ ,  $d_4$ , respectively, and then the maximum value  $d_0 = \max\{d_1, d_2, d_3, d_4\}$  is chosen as the radius of support domain. The dimension of the support domain is defined by  $d = \alpha_s d_0$ , where  $\alpha_s$  is generally chosen to be 1.5–3.0. In our examples in Section 5,  $\alpha_s = 2.0$  has been chosen and the number of points in the support domain are about 15–20 which leads to a reasonable small bandwidth for the system matrix.

# 3. HERMITE RADIAL BASIS POINT INTERPOLATION

The approximation of a function  $u(\mathbf{x})$  may be written as a linear combination of radial basis functions at the *n* nodes within support domain of  $\mathbf{x}$  and its normal derivatives at the  $n_b$  nodes on Neumann boundaries

$$u(\mathbf{x}) \cong \hat{u}(\mathbf{x}) = \sum_{i=1}^{n} a_i \phi_i + \sum_{j=1}^{n_b} b_j \frac{\partial \phi_j^b}{\partial \mathbf{n}} + G(\mathbf{x}), \qquad (11a)$$

$$\phi_i = \phi\left(\|\mathbf{x} - \mathbf{x}_i\|\right) \qquad \phi_j^b = \phi\left(\|\mathbf{x} - \mathbf{x}_j^b\|\right), \qquad \frac{\partial \phi_j^b}{\partial \mathbf{n}} = l_j^x \frac{\partial \phi_j^b}{\partial x} + l_j^y \frac{\partial \phi_j^b}{\partial y}.$$
 (11b)



: interior points and Dirichlet Boundary points
 : Neumann Boundary points
 : Collocation point
 NB : Neumann Boundary
 DB : Dirichlet Boundary

Figure 2. Hermite interpolation.

CONSTANT TERM.

$$G(\mathbf{x}) = g_0. \tag{12a}$$

LINEAR POLYNOMIAL.

$$G(\mathbf{x}) = g_0 + g_1 x + g_2 y.$$
 (12b)

Square Polynomial.

$$G(\mathbf{x}) = g_0 + g_1 x + g_2 y + g_3 x^2 + g_4 x y + g_5 y^2.$$
(12c)

 $a_i$  are coefficients which correspond to radial basis  $\phi_i$  of function,  $b_j$  are coefficients which correspond to normal derivative of radial basis  $\phi_j$  of function at the points on Neumann boundaries, and  $g_0, g_1, g_2, \ldots$  are the coefficients of the additional unknown polynomial.  $\phi$  is the radial basis.  $l_j^x, l_j^y$  are the elements of normal vector at the jth point on Neumann boundaries.

The coefficients  $a_i$  and  $b_j$  in equation (1) can be determined by enforcing that the function interpolations pass through all n nodes within the support domain and the normal derivatives' interpolations of function pass through  $n_b$  nodes on Neumann boundaries. Figure 2 is shown to demonstrate the idea of Hermite interpolation.

The interpolations of the function at the  $k^{\text{th}}$  point have the form:

$$\hat{u}_{k} = \hat{u}\left(\mathbf{x}_{k}\right) = \sum_{i=1}^{n} a_{i}\phi_{ki} + \sum_{j=1}^{n_{b}} b_{j}\frac{\partial\phi_{kj}^{b}}{\partial\mathbf{n}} + G(\mathbf{x}_{k}), \qquad k = 1, 2...n,$$
(13a)

$$\phi_{ki} = \phi\left(\left\|\mathbf{x}_{k} - \mathbf{x}_{i}\right\|\right), \qquad \phi_{kj}^{b} = \phi\left(\left\|\mathbf{x}_{k} - \mathbf{x}_{j}^{b}\right\|\right).$$
(13b)

The interpolations of the normal derivatives of function at the mth point on the Neumann boundaries have the form:

$$\frac{\partial \hat{u}_{m}^{b}}{\partial \mathbf{n}} = \frac{\partial \hat{u}^{b}\left(\mathbf{x}_{m}\right)}{\partial \mathbf{n}} = \sum_{i=1}^{n} a_{i} \frac{\partial \phi_{mi}}{\partial \mathbf{n}} + \sum_{j=1}^{n_{b}} b_{j} \frac{\partial}{\partial \mathbf{n}} \left(\frac{\partial \phi_{mj}^{b}}{\partial \mathbf{n}}\right) + \frac{\partial G\left(\mathbf{x}_{m}\right)}{\partial \mathbf{n}}, \qquad m = 1, 2, \dots n_{b}.$$
(14)

In addition, the additional polynomial terms have to satisfy an extra requirement that guarantees unique approximation of the function, and the following constraints are usually imposed:

$$\sum_{k=1}^{n} a_k = 0, \tag{15a}$$

$$\sum_{k=1}^{n} a_k x_k = 0, \qquad \sum_{k=1}^{n} a_k y_k = 0, \tag{15b}$$

$$\sum_{k=1}^{n} a_k x_k^2 = 0, \qquad \sum_{k=1}^{n} a_k x_k y_k = 0, \qquad \sum_{k=1}^{n} a_k y_k^2 = 0.$$
(15c)

For constant additional term, only one constraint equation (15a) is enforced (l = 1). For linear additional term, three constraint equations (15a),(15b) are enforced (l = 3). For quadratic additional terms, six constraint equations (15a)–(15c) are enforced (l = 6).

The interpolations of function and normal derivatives on the Neumann boundaries can be expressed by matrix formulations as follows:

$$\begin{cases} \hat{\mathbf{u}}_{(n+n_b)}^e \\ \mathbf{0}_{l\times 1} \end{cases} = \Psi_{(n+n_b+l)\times(n+n_b+l)} \mathbf{a}_{(n+n_b+l)\times 1},$$
 (16)

where  $\hat{\mathbf{u}}^e$  is the vector that collects all variables of the nodal function values at the *n* nodes in the support domain and all variables of normal derivatives of the nodal function at the  $n_b$  nodes on the Neumann boundaries in the support domain.

$$\hat{\mathbf{u}}^{e} = \begin{bmatrix} \hat{u}_{1} & \cdots & \hat{u}_{k} & \cdots & \hat{u}_{n} & \frac{\partial \hat{u}_{1}^{b}}{\partial \mathbf{n}} & \cdots & \frac{\partial \hat{u}_{m}^{b}}{\partial \mathbf{n}} & \cdots & \frac{\partial \hat{u}_{n_{b}}^{b}}{\partial \mathbf{n}} \end{bmatrix}_{1 \times (n+n_{b})}^{1}.$$
(17a)

The coefficients vector **a** is defined as

$$\mathbf{a} = \begin{bmatrix} a_1 & \cdots & a_i & \cdots & a_n & b_1 & \cdots & b_j & \cdots & b_{n_b} & g_0 & \cdots \end{bmatrix}_{1 \times (n+n_b+l)}^\top .$$
(17b)

The elements of  $\Psi$  are formed by  $\phi_{ki}$ ,  $\phi_{kj}$ 's normal derivatives and  $G(\mathbf{x})$ , and they can be obtained by equations (13a),(13b),(14), and (15a)-(15c).

Thus, the unknown coefficients vector

$$\mathbf{a} = \boldsymbol{\Psi}^{-1} \left\{ \begin{array}{c} \hat{\mathbf{u}}^{\boldsymbol{e}}_{(n+n_b)} \\ \mathbf{0}_{l\times 1} \end{array} \right\}.$$
(18)

Finally, the approximation form of function can be obtained as follows:

$$\hat{\mathbf{u}}(\mathbf{x}) = \phi a = \phi \Psi^{-1} \left\{ \begin{array}{c} \hat{\mathbf{u}}^{e}_{(n+n_b)} \\ \mathbf{0}_{l\times 1} \end{array} \right\} = \psi \hat{\mathbf{u}}^{e}.$$
(19a)

The matrix of radial basis, its normal derivatives and the additional terms is defined by

$$\boldsymbol{\phi} = \begin{bmatrix} \phi_1 & \cdots & \phi_i & \cdots & \phi_n & \frac{\partial \phi_1^b}{\partial n} & \cdots & \frac{\partial \phi_m^b}{\partial n} & \cdots & \frac{\partial \phi_{n_b}^b}{\partial n} & 1 & x & y \cdots \end{bmatrix}_{1 \times (n+n_b+l)}.$$
(19b)

The matrix of shape functions can be expressed as follows

$$\boldsymbol{\psi}_{(n+n_b)} = \begin{bmatrix} \psi_1 & \cdots & \psi_i & \cdots & \psi_n & \psi_1^H & \cdots & \psi_m^H & \cdots & \psi_{n_b}^H \end{bmatrix}.$$
(19c)

Here  $\psi_i$  (i = 1, 2, ..., n),  $\psi_j^H$   $(j = 1, 2, ..., n_b)$  are shape functions, and they are obtained by the first  $n + n_b$  elements in the vector  $[\phi \Psi^{-1}]_{1 \times (n+n_b+l)}$ .

Radial Point Interpolation Collocation Method

		0	•	0		
1-	D 3-r	ode (	Colloc	ation	Sche	me
	0	0	٠	0	0	
1	-D 5-r	node	Colloo	cation	Sche	me
0	0	0	•	0	0	0
1-	D 7-r	ode (	Colloc	ation	Sche	me

• : Collocation point

Figure 3. 1-D collocation scheme.







Finally, the function  $u(\mathbf{x})$  can be expressed as follows:

$$\hat{u} = \sum_{k=1}^{n} \psi_k \hat{u}_k^e + \sum_{j=1}^{n_b} \psi_j^H \frac{\partial \hat{u}_j^e}{\partial n}.$$
(20)

# 4. COLLOCATION SCHEMES

Consider the partial differential equation given by

$$L(u) = A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = H, \quad \text{in } \Omega,$$
(21a)

together with the general boundary.

NEUMANN BOUNDARY CONDITION.

$$L_{b1}(u) = \mathbf{n}^{\mathsf{T}} \cdot \nabla u + \bar{q}_n = 0, \quad \text{on } \Gamma_{b1}.$$
(21b)

DIRICHLET BOUNDARY CONDITION.

$$u - \bar{u} = 0, \qquad \text{on } \Gamma_{b2}. \tag{21c}$$

The coefficients A, B, C, D, E, F, and H may all depend upon x and y.

Assume that there are  $N_d$  internal (domain) points and  $N_b = N_{b1} + N_{b2}$  boundary points, where  $N_{b1}$  are Neumann boundary points and  $N_{b2}$  are Dirichlet boundary points.

In general, the location of the collocation points can be different from the location of nodes in the discretization model. However, for the sake of simplicity, collocation points are the same as the nodes of the model. Figure 3 shows the 1-D collocation schemes, and Figure 4 shows the 2-D collocation schemes. These collocation schemes are used in the computations of numerical examples in Section 5.

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The following  $N_d + N_{b1}$  equations are satisfied at internal domain nodes and Neumann boundary points:

$$L(\hat{u}_i) = A \frac{\partial^2 \hat{u}_i}{\partial x^2} + B \frac{\partial^2 \hat{u}_i}{\partial x \partial y} + C \frac{\partial^2 \hat{u}_i}{\partial y^2} + D \frac{\partial \hat{u}_i}{\partial x} + E \frac{\partial \hat{u}_i}{\partial y} + F \hat{u}_i = H_i, \quad \text{in } \Omega \text{ and on } \Gamma_{b1}.$$
(22a)

The following  $N_{b1}$  equations are satisfied on the Neumann boundary  $\Gamma_{b1}$ :

$$\mathbf{n}^{\top} \cdot \nabla \hat{u}_i + \bar{q}_n = 0, \qquad i = 1, 2, \dots, N_{b1}.$$
(22b)

The following  $N_{b2}$  equations are satisfied on the Dirichlet boundary  $\Gamma_{b2}$ :

$$\hat{u}_i - \bar{u} = 0, \qquad i = 1, 2, \dots, N_{b2}.$$
 (22c)

 $\hat{u}_i$  are obtained by equation (6) or (20). Its derivatives can be obtained by the following equations.

For radial point interpolation:

$$\hat{u}(\mathbf{x}) = \sum_{j=1}^{n} \psi_{j} \hat{u}_{j}^{e}, \qquad \frac{\partial \hat{u}(\mathbf{x})}{\partial x} = \sum_{j=1}^{n} \frac{\partial \psi_{j}}{\partial x} \hat{u}_{j}^{e}, \qquad \frac{\partial^{2} \hat{u}(\mathbf{x})}{\partial x^{2}} = \sum_{j=1}^{n} \frac{\partial^{2} \psi_{j}}{\partial x^{2}} \hat{u}_{j}^{e},$$

$$\frac{\partial \hat{u}(\mathbf{x})}{\partial y} = \sum_{j=1}^{n} \frac{\partial \psi_{j}}{\partial y} \hat{u}_{j}^{e}, \qquad \frac{\partial^{2} \hat{u}(\mathbf{x})}{\partial y^{2}} = \sum_{j=1}^{n} \frac{\partial^{2} \psi_{j}}{\partial y^{2}} \hat{u}_{j}^{e}, \qquad \frac{\partial^{2} \hat{u}(\mathbf{x})}{\partial x \partial y} = \sum_{j=1}^{n} \frac{\partial^{2} \psi_{j}}{\partial x \partial y} \hat{u}_{j}^{e}.$$
(23a)

For Hermite radial point interpolation:

$$\hat{u}(\mathbf{x}) = \sum_{k=1}^{n} \psi_k \hat{u}_k^e + \sum_{j=1}^{n_b} \psi_j^H \frac{\partial \hat{u}_j^e}{\partial n}, \qquad \qquad \frac{\partial \hat{u}(\mathbf{x})}{\partial x} = \sum_{k=1}^{n} \frac{\partial \psi_k}{\partial x} \hat{u}_k^e + \sum_{j=1}^{n_b} \frac{\partial \psi_j^H}{\partial x} \frac{\partial \hat{u}_j^e}{\partial n},$$

$$\frac{\partial^2 \hat{u}(\mathbf{x})}{\partial x^2} = \sum_{k=1}^{n} \frac{\partial^2 \psi_k}{\partial x^2} \hat{u}_k^e + \sum_{j=1}^{n_b} \frac{\partial^2 \psi_j^H}{\partial x^2} \frac{\partial \hat{u}_j^e}{\partial n}, \qquad \qquad \frac{\partial \hat{u}(\mathbf{x})}{\partial y} = \sum_{k=1}^{n} \frac{\partial \psi_k}{\partial y} \hat{u}_k^e + \sum_{j=1}^{n_b} \frac{\partial \psi_j^H}{\partial y} \frac{\partial \hat{u}_j^e}{\partial n}, \qquad (23b)$$

$$\frac{\partial^2 \hat{u}(\mathbf{x})}{\partial y^2} = \sum_{k=1}^{n} \frac{\partial^2 \psi_k}{\partial y^2} \hat{u}_k^e + \sum_{j=1}^{n_b} \frac{\partial^2 \psi_j^H}{\partial y^2} \frac{\partial \hat{u}_j^e}{\partial n}, \qquad \qquad \frac{\partial^2 \hat{u}(\mathbf{x})}{\partial x \partial y} = \sum_{k=1}^{n} \frac{\partial^2 \psi_k}{\partial x \partial y} \hat{u}_k^e + \sum_{j=1}^{n_b} \frac{\partial^2 \psi_j^H}{\partial x \partial y} \frac{\partial \hat{u}_j^e}{\partial n}.$$

Thus,  $\hat{u}_i$  and its derivatives in equation (22) can be obtained by substituting x into  $x_i$  in equation (23a) or (23b):

$$\hat{u}_{i} = \hat{u}(\mathbf{x}_{i}), \qquad \frac{\partial \hat{u}_{i}}{\partial x} = \frac{\partial \hat{u}(\mathbf{x}_{i})}{\partial x}, \qquad \frac{\partial^{2} \hat{u}_{i}}{\partial x^{2}} = \frac{\partial^{2} \hat{u}(\mathbf{x}_{i})}{\partial x^{2}},$$

$$\frac{\partial \hat{u}_{i}}{\partial y} = \frac{\partial \hat{u}(\mathbf{x}_{i})}{\partial y}, \qquad \frac{\partial^{2} \hat{u}_{i}}{\partial y^{2}} = \frac{\partial^{2} \hat{u}(\mathbf{x}_{i})}{\partial y^{2}}, \qquad \frac{\partial^{2} \hat{u}_{i}}{\partial x \partial y} = \frac{\partial^{2} \hat{u}(\mathbf{x}_{i})}{\partial x \partial y}.$$
(24)

# 5. NUMERICAL TESTS

In this section, a series of test examples are numerically analysed. 1-D examples for wave propagation and boundary layer problems are first examined to test the accuracy and the hconvergence rates of the proposed RPICM. The second and third examples involves solving 2-D Poisson equations with only Dirichlet boundary conditions. The results are obtained and compared with Gaussian RPICM based on different additional terms, namely no additional terms, constant term and linear terms. Several different results are obtained by using Gaussian and thin plate spline (TPS) RPICM. Their h-convergence rates are also investigated. For Gaussian RPICM, its computed results with different shape parameters are demonstrated. Examples 4 and 5 will be employed to study how the accuracy of solution for PDEs with Neumann boundary conditions can be improved. The Hermite-type point interpolation applied to deal with Neumann boundary conditions has shown very good improvement on the accuracy of solutions.

The error indicators used in Tables 1-12 and Figures 5-10 are defined as follows:

$$e = \sqrt{\frac{\sum_{i=1}^{N} (u_i^{ex} - \hat{u}_i)^2}{\sum_{i=1}^{N} (u_i^{ex})^2}}, \qquad e_x = \sqrt{\frac{\sum_{i=1}^{N} (u_{i,x}^{ex} - \hat{u}_{i,x})^2}{\sum_{i=1}^{N} (u_{i,x}^{ex})^2}}, \qquad e_y = \sqrt{\frac{\sum_{i=1}^{N} (u_{i,y}^{ex} - \hat{u}_{i,y})^2}{\sum_{i=1}^{N} (u_{i,y}^{ex})^2}}.$$
 (25a)

The rates of *h*-convergence of the relative error,  $\mathbf{R}(\eta)$ , are also computed in some examples. It is defined as follows:

$$\mathbf{R}(\eta) = \left| \frac{\log\left(\eta_i/\eta_{i+1}\right)}{\log\left(h_i/h_{i+1}\right)} \right|$$
(25b)

where  $\eta = e$ ,  $e_x$  or  $e_y$ , while  $h_{i+1}$  and  $h_i$  are the uniform nodal interval in the current and previous case, respectively.

### 5.1. 1-D

EXAMPLE 1. WAVE PROPAGATION PROBLEM. A one-dimensional example of Poisson equation will be analyzed in order to investigate h-convergence for RPICM. The governing equation and boundary conditions are

$$\frac{d^{2}u}{dx^{2}} + \lambda u = 0, \qquad x \in (0, 1) 
u(0) = 0, 
u(1) = 1.0,$$
(26)

where  $\lambda = 10.0$ .

The exact solution is:

$$u^{ex}(x) = \frac{\sin\sqrt{\lambda}x}{\sin\sqrt{\lambda}}.$$
(27)

As a first example, the 1-D equation is solved using Gaussian RPICM and thin plate spline (TPS) RPICM.

### • Gaussian RPICM

Three different forms of additional terms, namely no additional terms, constant additional terms and linear additional terms, have been utilised to investigate its accuracy.

Figure 5a shows the relative errors of function obtained by using Gaussian RPICM with different shape parameter values under the assumption of three different additional terms imposed when uniform 41-node model and five-node collocation scheme were adopted. Generally, no apparent improvement was observed when additional polynomial terms were employed. However, for some certain shape parameter c, a small improvement of accuracy can be obtained after using additional polynomial terms. With shape parameter c = 1.0, the  $L^2$  relative error with no additional term is 0.3%; the  $L^2$  relative error with constant additional term is 0.125%; the  $L^2$  relative error with linear additional term is 0.027%; From our calculated results, the relative errors of derivative is close to that of the function.

In this example, a uniform distribution of 21 (h = 0.05), 41 (h = 0.025), and 81 (h = 0.0125), points were also employed to study the *h*-convergence behaviour of the method. Figure 5b shows its *h*-convergence solutions with function and its derivative with different additional terms for shape parameter c = 1.0 when three-node collocation scheme was used. It is clear that no improvement was observed with different additional terms for convergence rates. However, the



(a) The errors for 41-node model (five-node collocation scheme). (b) The h-convergnce (three-node collocation scheme).

Figure 5. The results obtained with Gaussian RPICM for Example 1.

improvement of accuracy is apparent using additional terms. It should be noted that the test of the h-convergence failed for some shape parameters and collocation schemes.

# • TPS RPICM

The three-order and four-order TPS RPICM with additional quadratic polynomial has been employed to solve the 1-D example. The results of h-convergence have been listed in Tables 1 and 2 (for three-order TPS) and Tables 3 and 4 (for four-order TPS). From these results, it can be observed that good h-convergence rates have been obtained. In addition, the results show that the accuracy obtained by using four-order TPS RPICM with additional quadratic polynomial has further been improved.

Collocation Scheme	Five-No	de Scheme	Seven-Node Scheme		
Model Nodes	e(%)	R	e(%)	R	
21	7.622		6.315		
41	2.331	1.71	1.669	1.92	
81	0.615	1.92	0.417	2.00	

Table 1. *h*-convergence rates of u for Example 1 (M = 3).

Collocation Scheme	Five-Noc	le Scheme	Seven-Node Scheme		
Model Nodes	$e_x(\%)$	R	$e_x(\%)$	R	
21	7.553		6.300		
41	2.310	1.71	1.655	1.93	

Table 2. *h*-convergence rates of  $u_x$  for Example 1 (M = 3).

Table 3. *h*-convergence rates of u for Example 1 (M = 4).

0.609

1.92

0.413

2.00

81

Collocation Scheme	Five-No	de Scheme	Seven-Node Scheme		
Model Nodes	e(%)	R	e(%)	R	
21	10.84		0.956		
41	1.457	2.90	0.389	1.30	
81	0.217	2.75	0.050	2.96	

#### Radial Point Interpolation Collocation Method

Collocation Scheme	Five-Noc	le Scheme	Seven-Node Scheme		
Model Nodes	$e_x(\%)$	R	$e_x(\%)$	R	
21	10.67		1.032		
41	1.417	2.91	0.398	1.38	
81	0.207	2.78	0.052	2.94	

Table 4. *h*-convergence rates of  $u_x$  for Example 1 (M = 4).

# 5.2. 2-D Example: Elliptic Partial Differential Equations (PDEs)

EXAMPLE 2. POISSON EQUATION WITH UNIFORM DIRICHLET BOUNDARY CONDITION.

$$\nabla^2 u = \sin(\pi x) \sin(\pi y),$$
  $(x, y) \in \Omega = [0, 1] \times [0, 1],$  (28a)

$$u(x,y)|_{\partial\Omega} = 0. \tag{28b}$$

The exact solution is given by

 $u^{ex}(x,y) = -\frac{1}{2\pi^2}\sin(\pi x)\sin(\pi y).$  (29)



(a) The exact for Example 2.



Figure 6. The exact solutions for Example 2 and Example 3.





(b) The *h*-convergence.

Figure 7. The results obtained with Gussian RPICM for Example 2 (nine-node collocation scheme).

Table 5. The relative errors obtained with Gaussian RPICM (c = 1.0) for Example 2 (121-node regular model, nine-node collocation scheme).

No Additional Terms			Constant Additional Terms			Linear Additional Terms		
e(%)	$e_x(\%)$	$e_y(\%)$	$e(\%) e_x(\%) e_y(\%)$			e(%)	$e_x(\%)$	$e_y(\%)$
0.073	0.73	0.73	0.12	0.77	0.77	0.12	0.13	0.13

### Gaussian RPICM

The exact solution in equation (29) is shown in Figure 6a. Figure 7a shows the relative  $(L^2)$  and absolute  $(L^{\infty})$  errors of function and its derivatives with different shape parameter values when uniform  $11 \times 11$  model and nine-node collocation scheme were adopted. From these results shown as Figure 7b, three apparent optimal solutions are observed at c = 1.0, c = 3.0, and c = 10.0.

The *h*-convergence of this method using a uniform distribution of  $11 \times 11$  (h = 0.1),  $21 \times 21$  (h = 0.05), and  $41 \times 41$  (h = 0.025) nodes is shown in Figure 7b when nine-node collocation scheme was used. Similar to the 1-D example, the convergence rates about function and its derivatives are almost the same. The results from using  $11 \times 11$  uniformly distributed nodes model and nine-node collocation scheme using different additional terms are listed in Table 5. From Table 5, the improvement of accuracy, especially for the derivatives, was apparent using linear additional terms.

In order to investigate the suitability of this method for an irregular model, a 121-node scattered point model shown in Figure 8 is employed to solve this problem with different collocation schemes. These numerical results are listed in Table 6 for different collocation schemes and different additional terms. From Table 6, it is clear that the computed solution is close to the exact solution as we increase the nodal numbers in a collocation support domain. However, no improvement of accuracy can be observed with different additional terms.

EXAMPLE 3. POISSON EQUATION WITH NONUNIFORM DIRICHLET BOUNDARY CONDITION.

$$abla^2 u + u = (2+3x) e^{x-y}, \qquad (x,y) \in \Omega = [0,1] \times [0,1],$$
(30a)

$$u(x,y)|_{\partial\Omega} = (2+3x)e^{x-y}.$$
 (30b)

The exact solution is given by

$$u^{ex}(x,y) = xe^{x-y}.$$
 (31)





Radial Point Interpolation Collocation Method



Figure 8. Irregular model with 121 nodes.

Table 6. The relative errors obtained with Gaussian RPICM (c = 1.0) for Example 2 (121-node scattered points model).

	No Additional Terms				nt Additio	nal Terms	Linear Additional Terms		
$\alpha_s$	e(%)	$e_x(\%)$	$e_y(\%)$	e(%)	$e_x(\%)$	$e_y(\%)$	e(%)	$e_x(\%)$	$e_y(\%)$
1.0	5.73	19.73	42.84	5.70	18.31	34.79	19.36	34.48	85.38
1.5	0.30	0.71	1.30	0.285	3.45	1.55	0.32	3.30	1.40
2.0	0.035	0.093	0.08	0.037	0.159	0.184	0.036	0.18	0.21

# • Gaussian RPICM

The exact solution in equation (31) is shown in Figure 6b. Figure 9a shows the relative  $(L^2)$  and absolute  $(L^{\infty})$  errors of function and its derivatives with different shape parameter values when uniform  $11 \times 11$  model and nine-node collocation scheme were adopted. From these results shown in Figure 9a, only an apparent optimal solution can be observed at c = 0.02, and this is not the same as that obtained in the previous Example 2.

The *h*-convergence of this method using a uniform distribution of  $11 \times 11$  (h = 0.1),  $21 \times 21$  (h = 0.05) and  $41 \times 41$  (h = 0.025) points is shown in Figure 9b when nine-node collocation scheme is used.

In addition, this problem was solved using  $11 \times 11$  uniformly distributed points model and nine-node collocation scheme using different additional terms. These results are listed in Table 7. From Table 7, the improvement of accuracy, especially for the derivatives, is apparent using linear additional terms. This conclusion is similar to Example 2.

In order to investigate the suitability of this method for an irregular model, a 121-node scattered point model shown in Figure 8 was employed to solve this problem with different collocation schemes. These numerical results are listed in Table 8 for different collocation schemes and different additional terms. The same conclusions as in Example 2 can be obtained.

EXAMPLE 4. POISSON EQUATION WITH NEUMANN BOUNDARY CONDITION.

$$\nabla^2 u + u = (2+3x) e^{x-y}, \qquad (x,y) \in \Omega = [0,1] \times [0,1].$$
(32a)

Table 7. The relative errors obtained with Gaussian RPICM (c = 1.0) for Example 3 (121-node regular model, nine-node collocation scheme).

	No Additional Terms			Consta	nt Additiona	l Terms	Linear Additional Terms		
	e(%)	$e_x(\%)$	$e_y(\%)$	e(%)	$e_x(\%)$	$e_y(\%)$	e(%)	$e_x(\%)$	$e_y(\%)$
l	0.31	1.32	1.31	0.16	1.33	1.17	0.16	0.24	0.72

No Additional Terms				Constant Additional Terms			Linear Additional Terms		
as	e(%)	$e_x(\%)$	$e_y(\%)$	e(%)	$e_x(\%)$	$e_y(\%)$	e(%)	$e_x(\%)$	$e_y(\%)$
1.0	0.43	2.45	6.52	0.27	3.83	7.92	0.46	3.53	10.15
1.5	0.15	0.46	1.01	0.21	1.15	1.83	0.036	0.56	0.77
2.0	0.016	0.041	0.15	0.011	0.144	0.268	0.0088	0.086	0.22

Table 8. The relative errors obtained with Gaussian PICM (c = 1.0) for Example 3 (121-node scattered points model).

BOUNDARY CONDITION I.

$$u(x,y)|_{y=0} = xe^x, \qquad u(x,y)|_{y=1} = xe^{x-1}, \qquad \frac{\partial u}{\partial x}\Big|_{x=0} = e^{-y}, \qquad \frac{\partial u}{\partial x}\Big|_{x=1} = 2e^{1-y}.$$
 (32b)

BOUNDARY CONDITION II.

$$u(x,y)|_{x=0} = 0, \qquad u(x,y)|_{y=0} = xe^x, \qquad \frac{\partial u}{\partial x}\Big|_{x=1} = (1+x)e^{1-y}, \qquad \frac{\partial u}{\partial y}\Big|_{y=1} = -xe^{x-1}.$$
(32c)

The exact solution is given by

$$u^{ex}(x,y) = xe^{x-y}. (33)$$

### • Gaussian RPICM (c = 1.0)

The exact solution in equation (33) is shown in Figure 6b. This problem was solved using  $11 \times 11$ uniformly distributed points model and nine-node collocation scheme. The results obtained with two different interpolation scheme, namely directed collocation (DC) and Hermite interpolation collocation (HC), are listed in Table 9. From these results in Table 9, it is clear that HC schemes have greatly improved accuracy when there exist Neumann boundary conditions. The relative errors of function with DC and HC schemes are 8.47% and 3.30% respectively when Boundary Condition I was employed. The relative errors of function with DC and HC schemes are 20.08% and 0.30%, respectively, when Boundary Condition II was employed. A similar improvement of accuracy for derivatives can also be observed from Table 9. In addition, this problem was also solved using the 121-node scattered point model shown in Figure 8 to investigate the suitability of this method for an irregular model. These numerical results obtained with irregular model and different collocation schemes are listed in Table 10. These results show the efficiency of solving this problem using HC scheme even for the random scattered point model. The computed solution is close to the exact solution as we increase the nodal numbers in the collocation support domain for both DC scheme and HC scheme. The relative errors of function obtained with DC scheme are 14.98%, 3.15%, and 0.25%, respectively, when the sizes of support domain were chosen to be 1.0, 1.5, and 2.0. The relative errors of function obtained with HC scheme are 2.34%, 0.10%, and 0.03% respectively when the sizes of support domain were chosen to be 1.0, 1.5, and 2.0. A similar improvement of accuracy for derivatives can be observed from Table 10.

Table 9. The relative errors obtained with Gaussian RPICM (c = 1.0) for Example 4 (121-node regular model, nine-node collocation scheme).

	Boundar	y Conditio	n I	]	Boundary	Conditio	n II
	e(%)	$e_x(\%)$	$e_y(\%)$		e(%)	$e_x(\%)$	$e_y(\%)$
DC	8.47	8.53	36.92	DC	20.08	16.10	40.32
HC	3.30	2.77	9.65	нс	0.30	1.63	6.40

Table 10. The relative errors obtained with Gaussian RPICM (c = 1.0) for Example 4 (121-node scattered points model Boundary Condition II).

	Directed	l Collocati	on	Hermite Collocation				
αs	e(%)	$e_x(\%)$	$e_y(\%)$	as	e(%)	$e_x(\%)$	$e_y(\%)$	
1.0	14.98	11.77	28.80	1.0	2.34	3.31	13.57	
1.5	3.15	3.03	10.30	1.5	0.10	0.50	1.14	
2.0	0.25	0.23	0.63	2.0	0.03	0.14	0.20	

EXAMPLE 5. THE PARTIAL DIFFERENTIAL EQUATIONS WITH IRREGULAR SOLUTION DOMAIN.

$$\nabla \cdot (\mathbf{D}\nabla u) - \mathbf{v} \cdot \nabla u = f(x, y), \qquad (x, y) \in \Omega,$$
(34a)

$$\mathbf{D} = \begin{bmatrix} 1 & 0\\ 0 & (1+y^2) \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 1, (1+y)^2 \end{bmatrix}.$$
(34b)

DIRICHLET BOUNDARY CONDITIONS.

$$u|_{DB} = e^{1+y}. (35a)$$

NEUMANN BOUNDARY CONDITIONS.

$$\frac{\partial u}{\partial n}\Big|_{\substack{NB1\\NB2\\NB3}} = \frac{\partial}{\partial n} (e^{x+y} + (x^2 - x)^2 \log(1+y^2)) \Big|_{\substack{NB1\\NB2\\NB3}}.$$
(35b)

The exact solution is given by

$$u^{ex} = e^{x+y} + (x^2 - x)^2 \log(1 + y^2).$$
(36)

### • Gaussian RPICM (c = 6.0)

This is a problem with irregular solution domain shown in Figure 10a. It is solved using 81-node nonuniformly distributed point models shown in Figure 10b. The numerical results obtained with the nonuniform model and different collocation schemes are listed in Table 11. These results show the efficiency of solving this problem using HC scheme even for non-uniform models. The computed solution is closer to the exact solution as we increase the nodal numbers in the collocation support domain for both DC scheme and HC scheme. The relative errors of



Figure 10. Irregular solution domain for Example 5 and its 81-node discrete model.

	Directe	d Collocati	on	Hermite Collocation				
as	e(%)	$e_x(\%)$	$e_y(\%)$	$\alpha_s$	e(%)	$e_x(\%)$	$e_y(\%)$	
1.5	129.90	203.18	148.31	1.5	14.58	73.93	41.69	
2.0	38.37	61.32	58.12	2.0	6.14	12.36	15.68	
2.5	24.23	33.85	119.26	2.5	3.81	11.89	7.17	

Table 11. The relative errors obtained with Gaussian RPICM (c = 6.0) for Example 5 (81-node model, mixed boundary condition).

function obtained with DC scheme are 129.90%, 38.37%, and 24.23%, respectively, when the sizes of support domain were chosen to be 1.5, 2.0, and 2.5. The relative errors of function obtained with HC scheme are 14.58%, 6.14%, and 3.81%, respectively, when the sizes of support domain were chosen to be 1.5, 2.0, and 2.5. The similar improvement of accuracy for derivatives can be observed from Table 11.

# • TPS RPICM

TPS RPICM with different additional terms was applied to solve this problem. It is different from the Gaussian RPICM because it does not have the problem of an adjustable parameter. In order to avoid the singularity at r = 0 appearing from derivatives in our methods, at least a thin plate spline function of the order M = 2 should be adopted.

Directed Collocation				Hermite Collocation			
M = 4: Two-Order Thin Plate Spline with Constant Additional Terms							
$\alpha_s$	e(%)	$e_x(\%)$	$e_y(\%)$	$\alpha_s$	e(%)	$e_x(\%)$	$e_y(\%)$
1.5	6.70	21.90	38.16	1.5	> 100.00	> 200.00	> 200.00
2.0	59.78	77.26	111.16	2.0	5.67	50.36	13.39
2.5	15.04	58.54	> 200.00	2.5	5.61	13.07	5.41
M = 6: Three-Order Thin Plate Spline with Constant Additional Terms							
$\alpha_s$	e(%)	$e_x(\%)$	$e_y(\%)$	$\alpha_s$	e(%)	$e_x(\%)$	$e_y(\%)$
1.5	88.79	Too bad	Too bad	1.5	23.66	59.74	85.23
2.0	> 100.00	> 300.00	> 400.00	2.0	7.55	37.89	35.17
2.5	47.75	> 200.0	100.42	2.5	7.80	39.37	20.04
M = 4: Two-Order Thin Plate Spline with Linear Additional Terms							
$\alpha_s$	e(%)	$e_x(\%)$	$e_y(\%)$	as	e(%)	$e_x(\%)$	$e_y(\%)$
1.5	7.30	8.72	12.81	1.5	19.71	22.55	13.12
2.0	79.44	132.52	> 200.00	2.0	22.36	64.13	22.49
2.5	17.17	80.62	> 300.00	2.5	12.63	19.80	35.42
M = 6: Three-Order Thin Plate Spline with Linear Additional Terms							
$\alpha_s$	e(%)	$e_x(\%)$	$e_y(\%)$	$\alpha_s$	e(%)	$e_x(\%)$	$e_y(\%)$
1.5	14.94	29.34	38.96	1.5	1.29	6.76	3.76
2.0	2.16	> 100.00	12.66	2.0	1.91	3.52	3.66
2.5	0.88	9.46	5.58	2.5	1.52	2.84	2.47

Table 12. The relative errors obtained with thin plate spline RPICM for Example 5 (81-node model, mixed boundary condition).

Table 12 shows the relative errors of function and its derivatives for both DC scheme and HC scheme when 81-node model in which 23 are Neumann boundary points, nine are Dirichlet boundary points and the remaining 49 internal points (see Figure 10b) was used. From these results in Table 12, the relative errors obtained using TPS function without the additional terms are very high, and it shows that TPS function without the additional polynomial terms can not be adopted. However, the accuracy was greatly improved when using TPS function with linear polynomial term. The results in Table 12 show that relative errors of function with HC scheme are 1.29% when the size of support domain was chosen to be 1.5 and three-order TPS was used. In addition, these results still show that the higher order additional polynomial term must be added when high-order TPS was employed during the solution.

# 6. CONCLUSIONS

A point interpolation collocation method (PICM) based on radial basis is presented in this paper. In contrast to Galerkin-based approaches, the biggest advantage of this present method is its simplicity and its efficiency. Compared to radial basis function (RBF), its interpolation is implemented in a local support domain so that a banded system matrix will be acquired. In addition, the present method is the same as other point collocation methods: its implementation is straightforward, once the required derivatives are computed. Of course, the implementation of essential boundary conditions is straightforward in RPICM. This feature makes the RPICM truly meshfree and points can be sprinkled randomly for numerical analysis. A series of test examples were numerically analysed and some useful results have been obtained. An excellent scheme namely the Hermite-type interpolation was applied to greatly improve the accuracy when there exists Neumann boundary conditions. No major improvement on the accuracy of the results was observed when the additional polynomial terms were used for Gaussian radial basis. However an apparent improvement of accuracy can be obtained when the additional polynomial term was employed for high order TPS.

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